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The effect of interference of coloured additive and multiplicative white noises on escape rate*

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Abstract
In this paper we have calculated the escape rate from a meta stable state for coloured and correlated noise driven open systems based on the Fokker–Planck description of the stochastic process. We consider the effect of two correlation times due to the additive coloured noise and the correlation between additive coloured and multiplicative white noises. The effect of the noise correlation strength on the rate has also been investigated.

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1. Introduction

Ever since the seminal work of Kramers on the diffusion model of chemical reactions was published about half a century ago [1], the theory of activated processes has become a central issue in many areas of science [2, 3], notably in chemical physics, nonlinear optics and condensed matter physics. Kramers considered a model Brownian particle trapped in a one-dimensional well representing the reactant state which is separated by a barrier of finite height from a deeper well signifying the product state. The particle was supposed to be immersed in a medium such that the medium exerts a frictional force on the particle but at the same time thermally activates it so that the particle may gain enough energy to cross the barrier. Over several decades the model and many of its variants have served as standard paradigms in various problems of physical and chemical kinetics to understand the rate in multidimensional systems in the overdamped and underdamped limits [4–6], the effect of anharmonicities [6, 7], rate enhancement by parametric fluctuations [8], the role of non-Gaussian noise [7, 9–11], the role of a relaxing bath [12, 13], quantum and semiclassical corrections to classical rate and related aspects [14–21]. The vast body of literature has been the subject of several reviews [2, 3, 15] and monographs [17].

* This paper is dedicated to Professor D S Ray on the occasion of his 50th birthday.
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The common feature of an overwhelming majority of the aforesaid treatments is that the system is thermodynamically closed, which means that the noise of the medium is of internal origin so that the dissipation and fluctuation get related through the fluctuation–dissipation relation [22]. However, in a number of situations the system is thermodynamically open, i.e., the dissipation and the random force are not related through the fluctuation–dissipation relation [23]. In general, the origins of the noise in the open systems which exert two or more random forces are different. The barrier crossing dynamics with multiplicative and additive white noises aroused strong interest in the early eighties. Using a continued-fraction algorithm Faetti *et al* [24] successfully showed for the first time that the escape rate changes from the small value of the Kramers theory into the large relaxation rate of the Suzuki regime. It was shown, furthermore, that the time required to attain equilibrium in a well after sudden application of multiplicative noise (the activation time) is much shorter than Kramers' relaxation time.

In most of the work states above, noise forces that are present simultaneously in the stochastic systems were usually treated as random variables uncorrelated with each other. However, noises in some stochastic processes may have a common origin. If this happens, then the statistical properties of the noises should not be widely different and can be correlated. The cross correlated noises were first considered by Fedchenia [25] in the context of hydrodynamics of vortex flows in ellipsoidal containments with regard to fluctuations. Here the author introduced cross correlation among the noises of common origin which appear in the time evolution equation of dimensionless modes of flow rates. Fulinski and Telejko [26] also considered the interference of additive and multiplicative white noises in the bistable kinetic model, mentioning the physical possibility of cross correlated noises. However, very recently Madureira *et al* [27] have pointed out the possibility of cross correlated noise in the realistic model (ballast resistor) showing bistable behaviour and have also discussed the influence of correlation of additive and multiplicative white noises on the activated rate processes. The effect of correlation between additive and multiplicative noises is considered indispensable in explaining phenomena such as stochastic resonance, phase transition, transport in the superconducting junction and the transport of motor proteins etc [28–33]. Our aim in this paper is to investigate the effect of coupling between multiplicative and additive noise on the escape rate when the additive noise and the coupling between two noise terms are coloured with nonzero correlation times τ_1 and τ_2.

The outline of the paper is as follows. In section 2 we introduce the Fokker–Planck description of stochastic process and the calculation of the barrier crossing rate. The paper is concluded in section 3.

2. The Fokker–Planck description of noise driven process and the calculation of the barrier crossing rate

To begin with we consider noise driven dynamical systems in the overdamped limit. The Langevin equation of motion for the present problem can be written as

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{V'(q)}{\gamma} + \frac{q}{\gamma}\zeta(t) + \frac{1}{\gamma}\eta(t) \tag{1}$$

where $V'(q)$ is the derivative of the following double well potential with respect to the particle coordinate q, i.e.,

$$V(q) = -\tfrac{1}{2}aq^2 + \tfrac{1}{4}bq^4. \tag{2}$$

γ in equation (1) is the dissipation parameter. A schematic representation of the potential energy $V(q)$ is shown in figure 1. There $q = q_0$ and $q = q_b$ correspond to the bottom of the

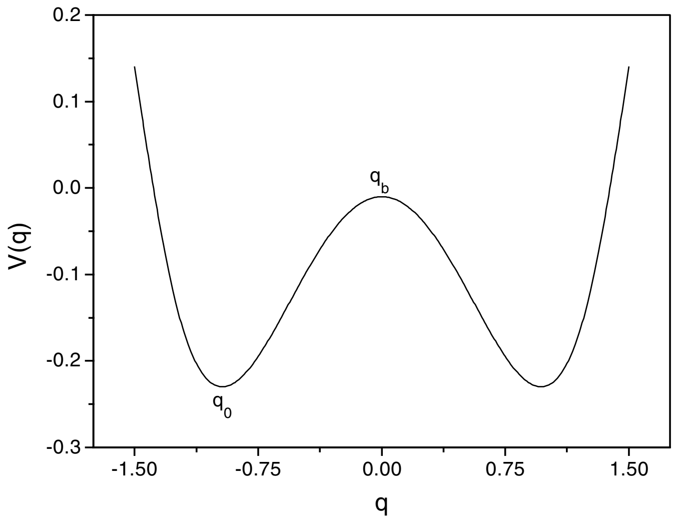

Figure 1. A schematic representation of the potential energy $V(q) = -\frac{a}{2}q^2 + \frac{1}{4}bq^4$.

left well and the barrier top of the potential, respectively. $\zeta(t)$ and $\eta(t)$ in equation (1) are white and coloured noises. The two noise terms are characterized by their mean and variance as

$$\langle\zeta(t)\rangle = \langle\eta(t)\rangle = 0 \tag{3}$$

$$\langle\zeta(t)\zeta(t')\rangle = 2D\delta(t-t') \tag{4}$$

$$\langle\eta(t)\eta(t')\rangle = \frac{D'}{\tau_1}\exp\left(-\frac{|t-t'|}{\tau_1}\right) \tag{5}$$

$$\langle\zeta(t)\eta(t')\rangle = \langle\eta(t)\zeta(t')\rangle = \frac{\lambda\sqrt{DD'}}{\tau_2}\exp\left(-\frac{|t-t'|}{\tau_2}\right). \tag{6}$$

Here τ_1 and D' are the correlation time and intensity of the coloured noise, respectively. D is the intensity of the white noise and τ_2 is the noise correlation time of the coupling between multiplicative and additive noises.

The first detailed analysis of the stochastic process with coloured noise was ventured by Sancho *et al* [34]. The authors derived a Fokker–Planck equation which is based on an expansion in terms of the noise correlation time. The work of Hängi *et al* [35] on the problem of barrier crossing driven by coloured noise, using non-perturbative analysis, shows very good agreement between numerical and analytical results for a small noise strength. Marchesoni [36] calculated the mean first passage time for the coloured noise driven systems using Kramers' approach. The papers [36, 37] are important critiques on this type of problem.

The random process with correlated coloured and white noise has been studied very recently by Luo *et al* [30]. Here the authors have derived a Fokker–Planck equation applying the Novikov theorem, Fox's approach and the unified coloured noise approximation method. Following [30] one can write the Fokker–Planck equation corresponding to the Langevin equation (1) with the noise properties (3)–(6) as

$$\frac{\partial\rho}{\partial t} = \left[\frac{\partial}{\partial q}\frac{V'(q)}{A(q)\gamma} - \frac{\partial}{\partial q}\left(g(q)\frac{\partial g(q)}{\partial q}\right) + \frac{\partial^2 g(q)^2}{\partial q^2}\right]\rho \tag{7}$$

where

$$A(q) = 1 + V''(q)\tau_1/\gamma \tag{8}$$

and

$$g(q) = \frac{\left[D' + \frac{2\lambda\sqrt{DD'}}{1+2\tau_2}q + Dq^2\right]^{1/2}}{\gamma A}. \tag{9}$$

The above Fokker–Planck equation takes the form

$$\frac{\partial\rho}{\partial t} = \left[\frac{\partial V'(q)}{\partial q} - \frac{\partial Dq}{\partial q} + \frac{\partial^2(D' + Dq^2)}{\partial q^2}\right]\rho \tag{10}$$

in the limit $\lambda = 0$, $\gamma = 1.0$ and $\tau_1 = 0.0$. It bears an exact resemblance to the Fokker–Planck equation developed earlier by Gammaitoni *et al* [38] for multiplicative and additive white noises.

Now taking the contribution of the linearized form of the potential $V(q)$ in the expression for A, the Fokker–Planck equation (7) can be written as

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial q}\frac{V'(q)\rho}{A\gamma} + \frac{\partial lq\rho}{\partial q} + \frac{\partial l_1\rho}{\partial q} + Q\frac{\partial^2\rho}{\partial q^2} - \frac{2D\rho}{A^2\gamma^2} \tag{11}$$

where

$$A = 1 + Z\tau_1/\gamma \tag{12}$$

$$l = \frac{3D}{\gamma^2 A^2} \tag{13}$$

$$l_1 = \frac{3\lambda\sqrt{DD'}}{\gamma^2 A^2(1+2\tau_2)} \tag{14}$$

and

$$Q = \frac{D' + \frac{2\lambda\sqrt{DD'}}{1+2\tau_2}q_e + Dq_e^2}{\gamma^2\left(1 + \frac{\omega_0^2\tau_1}{\gamma}\right)^2}. \tag{15}$$

Z in equation (12) is the value of curvature of the linearized potential around the fixed point and q_e in equation (15) is the solution of the algebraic equation

$$V'(q_e) + lq_e + l_1 = 0. \tag{16}$$

Thus q_e is the equilibrium value of q in the absence of Q [33]. ω_0 in equation (15) is the linearized frequency of the potential around q_0.

Now multiplying $\exp\left(\frac{2Dt}{A^2\gamma^2}\right)$ on both sides of the Fokker–Planck equation (11) followed by the transformation

$$W(q,t) = \rho(q,t)\exp\left(\frac{2Dt}{A^2\gamma^2}\right) \tag{17}$$

we get

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial q}\frac{V'(q)W}{A\gamma} + \frac{\partial lqW}{\partial q} + \frac{\partial l_1 W}{\partial q} + Q\frac{\partial^2 W}{\partial q^2}. \tag{18}$$

Recasting the above equation in the form of a continuity equation we identify j as the current

$$\begin{aligned}\frac{\partial W}{\partial t} &= -\frac{\partial}{\partial q}\left[-\frac{V'(q)W}{A\gamma} - lqW - l_1W - Q\frac{\partial W}{\partial q}\right]\\ &= -\frac{\partial}{\partial q}j(q,t).\end{aligned} \tag{19}$$

In the stationary state $j =$ constant, i.e., $\frac{\partial W}{\partial t} = 0$. j looks like

$$j = -\frac{V'(q)W}{A\gamma} - lqW - l_1 W - Q\frac{\partial W}{\partial q}. \tag{20}$$

Rearranging the above equation as

$$\frac{\partial W}{\partial q} + \frac{V'(q)W}{A\gamma Q} + \frac{lq}{Q}W + \frac{l_1}{Q}W = -\frac{j}{Q} \tag{21}$$

and then integrating between q_0 and B with the integrating factor $\exp\left(\int \frac{V'(q)/(A\gamma)+lq+l_1}{Q}\,\mathrm{d}q\right)$, in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[W(q)\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right)\right]$$
$$= -\frac{j}{Q}\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right) \tag{22}$$

we obtain

$$\left[W(q)\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right)\right]_{q_0}^{B}$$
$$= -\frac{j}{Q}\int_{q_0}^{B}\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right)\mathrm{d}q. \tag{23}$$

The constant current or flux across q_b is thus,

$$j = -Q\frac{\left[W(q)\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right)\right]_{q_0}^{B}}{\int_{q_0}^{B}\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right)\mathrm{d}q}. \tag{24}$$

Since the value of $W(q)$ at B is zero, i.e., $W(B) = 0$, we further obtain,

$$j = Q\frac{W(q_0)\exp\left(\frac{V(q_0)/(A\gamma)+lq_0^2/2+l_1q_0}{Q}\right)}{\int_{q_0}^{B}\exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right)\mathrm{d}q}. \tag{25}$$

We are now in a position to calculate the population in the left well at zero current. For $j = 0$, equation (20) becomes

$$\frac{\partial W}{\partial q} = -\frac{\frac{V'(q)}{A\gamma}+lq+l_1}{Q}W. \tag{26}$$

Integrating the above equation from q_0 to q (an arbitrary point in the left well) we obtain

$$W(q) = W(q_0)\exp\left(\frac{\left[V(q_0)/(A\gamma)+lq_0^2/2+l_1q_0\right]-\left[V(q)/(A\gamma)+lq^2/2+l_1q\right]}{Q}\right). \tag{27}$$

Thus the population in the left well at $j = 0$ is given by

$$n_a = \int_{q_1}^{q_2} W(q_0)\exp\left(\frac{\left[V(q_0)/(A\gamma)+lq_0^2/2+l_1q_0\right]-\left[V(q)/(A\gamma)+lq^2/2+l_1q\right]}{Q}\right)\mathrm{d}q \tag{28}$$

where q_1 and q_2 are two points around q_0.

The rate of escape k is given by

$$k = \frac{j}{n_a} \tag{29}$$

which we thus obtain from (25) and (28) as

$$k = \frac{Q}{\int_{q_0}^{B} \exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right) \mathrm{d}q \int_{q_1}^{q_2} \exp\left(-\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right) \mathrm{d}q}. \tag{30}$$

We now make use of the following linearization of $V(q)$ around q_0 and q_b. For the integral

$$\int_{q_0}^{B} \exp\left(\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right) \mathrm{d}q \tag{31}$$

we write $V(q)$ as

$$V(q) = V(q_b) - \tfrac{1}{2}\omega_b^2(q-q_b)^2 \tag{32}$$

and let $q_0 \to -\infty$ and $B \to +\infty$. Then around q_b, the forms of A, l and l_1 become

$$A_b = 1 - \omega_b^2\tau_1/\gamma \tag{33}$$

$$l_b = \frac{3D}{\gamma^2 A_b^2} \tag{34}$$

and

$$l_{1b} = \frac{3\lambda\sqrt{DD'}}{\gamma^2 A_b^2(1+2\tau_2)}. \tag{35}$$

Similarly for the integral

$$\int_{q_1}^{q_2} \exp\left(-\frac{V(q)/(A\gamma)+lq^2/2+l_1q}{Q}\right) \mathrm{d}q \tag{36}$$

we use

$$V(q) = V(q_0) + \tfrac{1}{2}\omega_0^2(q-q_0)^2 \tag{37}$$

and let $q_1 \to -\infty$, $q_2 \to +\infty$. The forms of A, l and l_1 around q_0 look like

$$A_0 = 1 + \omega_0^2\tau_1/\gamma \tag{38}$$

$$l_0 = \frac{3D}{\gamma^2 A_0^2} \tag{39}$$

and

$$l_{10} = \frac{3\lambda\sqrt{DD'}}{\gamma^2 A_0^2(1+2\tau_2)}. \tag{40}$$

Using equations (32) and (37) in the expression for k in equation (30) we obtain

$$k = \frac{1}{\pi\gamma}\sqrt{\frac{a_0 a_b}{A_0 A_b}} \exp\left(-\frac{E_0}{A_0\gamma Q}\right) \exp\left(\frac{6a_b\omega_b^2 E_0 - b_b^2}{16a_b A_b\gamma Q}\right) \exp\left(-\frac{b_0^2}{4a_0\gamma A_0 Q}\right) \tag{41}$$

where

$$a_0 = \tfrac{1}{2}\omega_0^2 + A_0 l_0\gamma/2 \tag{42}$$

$$b_0 = l_{10}\gamma A_0 \tag{43}$$

$$a_b = \tfrac{1}{2}\omega_b^2 - A_b l_b\gamma/2 \tag{44}$$

$$b_b = l_{1b}\gamma A_b + \omega_b^2\sqrt{3E_0}. \tag{45}$$

Here E_0 is the activation energy $E_0 = V(q_b) - V(q_0 = 0)$. The above expression (41) is the Kramers' rate for the coloured and correlated noise driven system. We are now in a position to check the several limits of the rate expression (41). First, we consider $\lambda = 0$ and $D = 0$. Then equation (41) takes the form

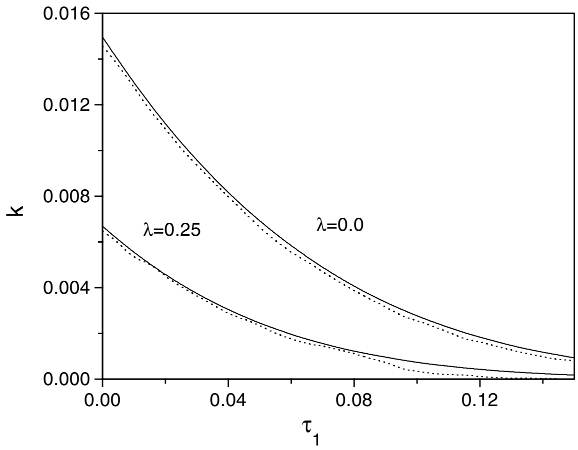

Figure 2. Plot of the rate constant k versus the noise correlation time τ_1 for $a = 1.0$, $b = 1.0$ and $D = D' = 0.05$.

$$k = \frac{\sqrt{2}a}{\pi\gamma}\sqrt{\frac{1}{A_0 A_b}}\exp\left(-\frac{E_0(1+2a\tau_1)}{\gamma D'}\right). \tag{46}$$

However, the above rate expression reduces to the following form in the limit $a\tau_1 < 1$

$$k = \frac{\sqrt{2}a}{\pi\gamma}\sqrt{\frac{1}{1+a\tau_1}}\exp\left(-\frac{E_0(1+2a\tau_1)}{\gamma D'}\right). \tag{47}$$

The pre-exponential factor in the above expression resembles that of the earlier result developed by Hängi *et al* [35] and the exponential factor in the above expression shows similarity with the result developed by Marchesoni [36].

One can now check that the above result (41) reduces to the standard result of the escape problem in the overdamped limit. Let us consider that the multiplicative noise strength D is zero and the dissipation parameter γ and the additive noise are related through the fluctuation–dissipation relation by

$$\langle \eta(t)\eta(t')\rangle = 2\gamma k_b T \delta(t-t') \tag{48}$$

in the Markovian limit. Here k_b is the Boltzmann constant and T is the temperature of the thermal bath. Under these conditions we have

$$A_0 = A_b = 1 \tag{49}$$

$$a_0 = \tfrac{1}{2}\omega_0^2 \qquad b_0 = 0 \qquad a_b = \tfrac{1}{2}\omega_b^2 \qquad b_b = \omega_b^2\sqrt{3E_0} \tag{50}$$

and

$$Q = \frac{k_b T}{\gamma}. \tag{51}$$

Putting the above equations (49)–(51) in equation (41) we obtain the standard result [2] for the escape rate as

$$k = \frac{\omega_0\omega_b}{2\pi\gamma}\exp\left(-\frac{E_0}{k_b T}\right). \tag{52}$$

Finally to verify the proposed theoretical result (41) we calculate the mean first passage time solving the Langevin equation (1) employing Heun's algorithm. In figure 2 we plot the rate constant against the correlation time of the additive coloured noise for both $\lambda = 0$ and $\lambda \neq 0$. The solid and dotted curves correspond to the analytical and numerical results, respectively. Thus theoretical estimation of the effect of cross correlation of the noises on the rate constant shows fair agreement with the numerical results.

3. Conclusions

In summary, the phenomenon of barrier crossing in the noise driven dynamical system is investigated in the present paper, when both the additive noise and the coupling between additive and multiplicative noise are coloured with different values of noise correlation times τ_1 and τ_2, based on the Fokker–Planck description of the stochastic process. Here we consider how the rate is affected by several factors such as the correlation time of the additive coloured noise, the correlation time due to coupling between additive and multiplicative noises, the strength of the noise correlation and dissipation. We have checked that our result reduces to the standard result when the multiplicative noise strength becomes zero and additive noises are related to the dissipation through the fluctuation–dissipation relation in the Markovian limit. Since in certain situations [24–31] both the multiplicative and additive noise may have a common origin and may be coupled as well, we hope that our present calculation is likely to be important in the interpretation of experimental findings.

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